# Chapter 17
## THE INTERNAL RATE OF RETURN

*A whimsical "nugget"*

If net present value (NPV) is inversely proportional to the discounting rate, then there must exist a discounting rate that makes NPV equal to zero.

The discounting rate that makes net present value equal to zero is called the "internal rate of return (IRR)" or "yield to maturity".

**To apply this concept to capital expenditure, simply replace "yield to maturity" by "IRR", as the two terms mean the same thing. It is just that one is applied to  nancial securities (yield to maturity) and the other to capital expenditure (IRR)**.

## Section 17.1
### HOW IS INTERNAL RATE OF RETURN DETERMINED?

To calculate IRR, make $r$ the unknown and simply use the NPV formula again. The rate $r$ is determined as follows:

$$VAN = 0, \text{ or } \sum_{n=1}^{N} \frac{F_n}{(1+r)^n} = V_0$$

To use the same example from the previous chapter:

$$\frac{0.8}{(1+r)} + \frac{0.8}{(1+r)^2} + \ldots + \frac{0.8}{(1+r)^5} = 2$$

In other words, an investment's internal rate of return is the rate at which its market value is equal to the present value of the investment's future cash  ows.

It is possible to use trial-and-error to determine IRR. This will result in an interest rate that gives a negative net present value and another that gives a positive net present value. These negative and positive values constitute a range of values, which can be narrowed until the yield to maturity is found, which in this case is about 28.6%.

Obviously, this type of calculation is time consuming. It is much easier to just use a calculator or spreadsheet program with a function to determine the yield to maturity.

## Section 17.2
## INTERNAL RATE OF RETURN AS AN INVESTMENT CRITERION

Internal rate of return is frequently used in nancial markets because it immediately tells the investor the return to be expected for a given level of risk. The investor can then compare this expected return to his required return rate, thereby simplifying the investment decision.

**The decision making rule is very simple: if an investment's internal rate of return is higher than the investor's required return, he will make the investment or buy the security. Otherwise, he will abandon the investment or sell the security.**

In our example, since the internal rate of return (28.6%) is higher than the return demanded by the investor (20%), he should make the investment. If the market value of the same investment were 3 (and not 2), the internal rate of return would be 10.4%, and he should not invest.

An investment is worth making when its internal rate of return is equal to or greater than the investor's required return. An investment is not worth making when its internal rate of return is below the investor's required return.

**Hence, at fair value, the internal rate of return is identical to the market return. In other words, net present value is nil.**

## Section 17.3
## THE LIMITS OF THE INTERNAL RATE OF RETURN

With this new investment decision-making criterion, it is now necessary to consider how IRR can be used vis-à-vis net present value. It is also important to investigate whether or not these two criteria could somehow produce contradictory conclusions.

If it is a simple matter of whether or not to buy into a given investment, or whether or not to invest in a project, the two criteria produce exactly the same result, as shown in the example.

If the cash ow schedule is the same, then calculating the NPV by choosing the discounting rate and calculating the internal rate of return (and comparing it with the discounting rate) are two sides of the same mathematical coin.

### 1/ THE REINVESTMENT RATE AND THE MODIFIED IRR (MIRR)

Consider two investments *A* and *B*, with the following cash ows:

| Year | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| Investment *A* | 6 | 0.5 | | | | | |
| Investment *B* | 2 | 3 | 0 | 0 | 2.1 | 0 | 5.1 |

At a 5% discount rate, the present value of investment *A* is 6.17 and that of investment *B* 9.90. If investment *A*'s market value is 5, its net present value is 1.17. If investment *B*'s market value is 7.5, its net present value is 2.40.

Now calculate their yield to maturity. It is 27.8% for investment *A* and 12.7% for investment *B*. Or, to sum up:

| | NPV at 5% | IRR% |
|---|---|---|
| Investment *A* | 1.17 | 27.8 |
| Investment *B* | 2.40 | 12.7 |

Investment *A* delivers a rate of return that is much higher than the required return (27.8% vs. 5%) during a short period of time. Investment *B*'s rate of return is much lower (12.7% vs. 27.8%), but is still higher than the 5% required return demanded and is delivered over a far longer period (seven years vs. two). Our NPV and internal rate of return models are telling us two different things. So should we buy investment *A* or investment *B*?

At rst glance, investment *B* would appear to be the more attractive of the two. Its NPV is higher and it creates the most value: 2.40 vs. 1.17.

However, some might say that investment *A* is more attractive, as cash ows are received earlier than with investment *B* and therefore can be reinvested sooner in high-return projects. While that is theoretically possible, it is the strong (and optimistic) form of the theory because competition among investors, and the mechanisms of arbitrage, tend to move net present values towards zero. Net present values moving towards zero means that exceptional rates of return converge toward the required rate of return, thereby eliminating the possibility of long-lasting high-return projects.

Given the convergence of the exceptional rates toward required rates of return, it is more reasonable to suppose that cash ows from investment *A* will be reinvested at the required rate of return of 5%. The exceptional rate of 27.8% is unlikely to be recurrent. And this is exactly what happens if we adopt the NPV decision rule. The NPV in fact assumes that the reinvestment of interim cash ows is made at the required rate of return ($k$):

$$\left[\sum_{n=1}^{N} F_n \times (1+k)^{N-n}\right] \times (1+k)^{-N} - F_0 = \sum_{n=1}^{N} \frac{F_n}{(1+k)^n} - F_0$$

If we apply the same equation to the IRR, we observe that the reinvestment rate is simply the IRR again. However, in equilibrium, it is unreasonable to think that the company can continue to invest at the same rate of the (sometimes) exceptional IRR of a speci c project. Instead it is much more reasonable to assume that, at best, the company can invest at the required rate of return.

However, a solution to the reinvestment rate problem of IRR is the **Modi ed IRR (MIRR)**.

MIRR is the rate of return that yields an NPV of zero when the initial outlay is compared with the terminal value of the project's net cash flows reinvested at the required rate of return.

Determining the MIRR requires two stages:

1 calculate forward until the end of the project to determine the terminal value of the project by compounding all intermediate cash  ows at the required rate of return; and
2  nd the internal rate of return that equates the terminal value with the initial outlay.

So by capitalising cash  ow from investments *A* and *B* at the required rate of return (5%) up to period 7, we obtain from investment A in period 7: $6 \times 1.005^6 + 0.5 \times 1.05^5$, or 8.68. From investment *B* we obtain $2 \times 1.05^6 + 3 \times 1.05^5 + 2.1 \times 1.05^2 + 5.1$, or 13.9. The internal rate of return is 8.20% for investment *A* and 9.24% for investment *B*.

We have thus reconciled the NPV and internal rate of return models.

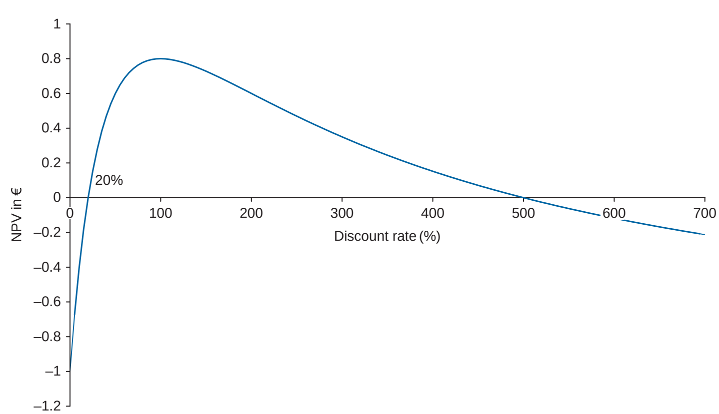
Some might say that it is not consistent to expect investment *A* to create more value than investment *B*, as only 5 has been invested in *A* vs. 7.5 for *B*. Even if we could buy an additional "half-share" of *A*, in order to equalise the purchase price, the NPV of our new investment in *A* would only be $1.17 \times 1.5 = 1.76$, which would still be less than investment *B*'s NPV of 2.40. For the reasons discussed above, we are unlikely to  nd another investment with a return identical to that of investment *A*.

Instead, we should assume that the 2.5 in additional investment would produce the required rate of return (5%) for seven years. In this case, NPV would remain by de nition at 1.17, whereas the internal rate of return of this investment would fall to 11%. NPV and the internal rate of return would once again lead us to conclude that investment *B* is the more attractive investment.

## 2/ MULTIPLE OR NO IRR

Consider the following investment:

| Year | 0 | 1 | 2 |
|---|---|---|---|
| Cash flow | −1 | 7.2 | −7.2 |




There are two annual rates of return! Which one should we choose? At 10%, the NPV of this investment is − 0.40. So it is not worth realising, even though its internal rate of return is higher than the required rate of return.

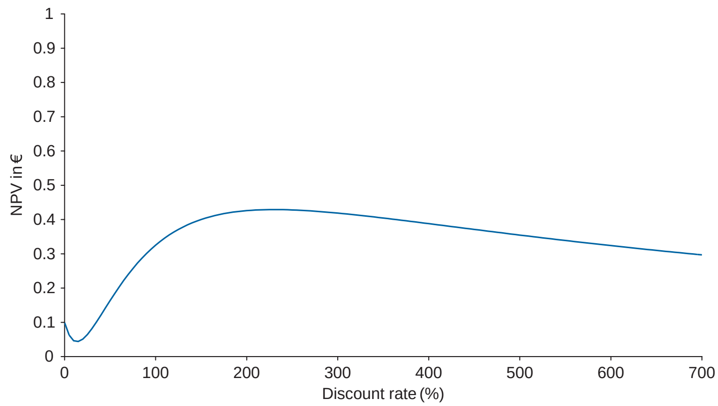
The project has two IRRs, and we do not know which is the right one. There is no good reason to use one over the other. Investments with “unconventional” cash ow sequences are rare, but they can happen. Consider a rm that is cutting timber in a forest. The timber is cut, sold and the rm gets an immediate pro t. But, when harvesting is complete, the rm may be forced to replant the forest at considerable expense.

Another example may be a strip-mining project, which normally requires a nal investment to reclaim the land and satisfy the requirements of environmental legislation.

Consider now the following investment:

| Year | 0 | 1 | 2 |
|---|---|---|---|
| Cash flow | 3.2 | −7.1 | 4.0 |

There is no internal rate of return that makes NPV zero! At 10%, the NPV of this investment is 0.05 and it is worth buying.




A project like this has no IRR. Thus, we have no benchmark for deciding if it is a good investment or not. Although the NPV remains positive for all the discount rates, it remains only *slightly* positive and the company may decide not to do it.

## 3/ INVESTING OR FINANCING?

Consider two projects with the following ows:

| Project | $F_0$ | $F_1$ | IRR | NPV (15%) |
|---|---|---|---|---|
| *A* | −100 | 120 | 20% | € 4.35 |
| *B* | 100 | −120 | 20% | −€ 4.35 |

The ows are exactly the same but with opposite signs. The IRR of the two projects is the same (20%) but the NPV is positive for project *A* and negative for project *B* (both with a discount rate of 15%). According to the IRR rule, project *A* and *B* have the same value; however, the NPV says that project *A* is preferable to project *B*.

Although an investment project with the cash ows of *B* may seem quite unusual, there are some situations where it is possible. For example, consider a business school conducting seminars and courses whereby the participants pay in advance. Large expenses (travelling expenses of external teachers, materials and salaries of teachers, etc.) are incurred at the seminar date or later on: thus cash in ows precede cash out ows.

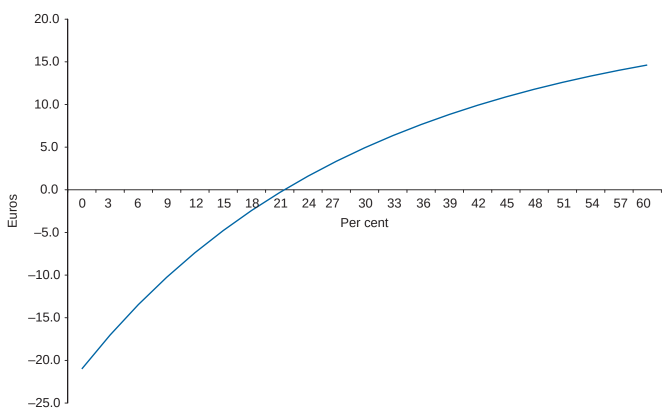
Consider our trial-and-error method to calculate the IRR of project *B*:

| $F_0$ | $F_1$ | $k$ | NPV |
|---|---|---|---|
| 100 | −120 | 15% | −€ 4.35 |
| 100 | −120 | 20% | € 0.00 |
| 100 | −120 | 30% | € 7.69 |

The reader will surely have noticed that the net present value of project*B* is *negative* when the discount rate is below 20%. Conversely, the NPV is positive when the discount rate is above 20%.

The decision rule for this kind of project is exactly the opposite to the “traditional” IRR rule. In fact, you should **accept the project when IRR is less than the discount rate and reject the project when IRR is greater than the discount rate.**

Why has the rule ended up being inverted like this? The reason is clearly shown in the graph of the NPV pro le of project *B*. The curve is upward sloping (similar to a loan), implying that NPV is *positively related* to the discount rate.




Intuitively, the “inverted IRR” rule makes sense. If the rm wants to obtain €100 immediately, it can either invest in project B or *borrow €100* from a bank, which will have to be repaid in the following period with an interest rate of 20%. Thus, the project is actually a substitute for borrowing.

## 4/ CHANGING DISCOUNT RATES

It is common to discount cash ows at a constant rate throughout a project’s life. However, this may not be appropriate under certain circumstances. In fact, the required rate of return is a function of interest rates and of the uncertainty of cash ows, both of which can change substantially over time.

The necessity of using different discount rates can be easily overcome with the NPV criteria, whereby different discount rates can be set for each period. Conversely, the IRR method can only be compared with a single rate of return and cannot cope with changing discount rates.

## 5/ PROBLEMS SPECIFIC TO MUTUALLY EXCLUSIVE PROJECTS

Further problems may arise when a choice must be made among several investments (or securities), as is often the case in reality. Investments have different cash  ow timetables that are all equally attractive. In this case, the investment decision is not about whether to invest or not, but rather it is about *which* investment to make. This situation refers to **mutually exclusive investments**. This occurs when there are two projects, *A* and *B*, and you can either accept *A*, accept *B*, or reject both projects, *but it is impossible to accept both of them simultaneously.*

Why would a company decide to abandon one or more viable projects? Typically, the dilemma arises from **capital rationing**. Capital rationing may arise for two reasons:

- because a  rm cannot obtain funds at market rates of return (**hard rationing** ). Hard rationing implies market imperfections, transaction costs and agency costs arising from the separation of ownership and management; or
- because of internally imposed  nancial constraints by management (**soft rationing**). Soft rationing may arise when:
    - there are maximum limits on borrowing and shareholders are reluctant to inject additional equity;
    - the management intends to pursue a steady-growth strategy, avoiding exceptional growth rates; and
    - there are divisional ceilings imposed through annual capital budgets.

Mutually exclusive projects may give rise to two problems: the “scale problem” and the “timing problem”, both of which will be examined next.

To understand the **scale problem**, consider two projects of different dimensions, one of which can be de ned as a small-scale project, and the other as a large-scale project:

| Project | $F_0$ | $F_1$ | IRR | NPV (10%) |
|---|---|---|---|---|
| Small-scale | −10 | 15 | 50% | € 3.64 |
| Large-scale | −100 | 120 | 20% | € 9.09 |

The point of this example is that when considering two mutually exclusive investments, the  nancial manager typically concludes that the one offering the highest IRR is necessarily the one that should be chosen. If in this case we had to choose only one project, and we rank them based on their IRRs, we would choose to invest in the small-scale project. However, the large-scale project generates a much higher NPV; this project thus creates more wealth for shareholders. The NPV tells us to undertake the large-scale project.

Why is there this con ict? The large-scale project is 10 times bigger than the small-scale project. Even though the latter provides a higher rate of return, the opportunity of making a much larger investment seems more attractive for shareholders.

For managers who prefer to use the IRR method, there is a solution to the scale problem. The approach is to calculate the IRR for an imaginary project with cash flows equal to the difference in cash flows between the large-scale and small-scale investments. This difference is defined as the *incremental project*.

The financial manager can use the incremental project's cash flows to determine the **incremental IRR**, i.e. the incremental return from choosing the large project instead of the small project:

| | $F_0$ | $F_1$ | **Incremental IRR** | NPV (10%) |
|---|---|---|---|---|
| Incremental (Large- to small-scale) | −90 | 105 | **17%** | € 5.45 |

If, as in this example, the incremental project's IRR is higher than the required rate of return, then the large-scale investment is better. If the inverse is true, then we should accept the small-scale project.

The logic of this approach works because both projects exceed the required rate of return. Therefore, this method is like equating the bigger-scale project to be the sum of the small-scale project and the incremental project. Then it is possible to examine the incremental project's cash IRR, and if it also exceeds the required rate of return, we can accept the bigger project. If not, then we should opt for the small-scale project.

Why is this? If we accept the large-scale investment we are in fact making two investments, not just one. We are accepting one project with cash flows identical to those of the small-scale project and another with cash flows equal to those of the incremental project. Since both projects (small-scale and incremental) exceed the required rate of return, we may conclude that we are happy to undertake the incremental project *and* the small-scale project. The only way to do both is to accept the large-scale project.

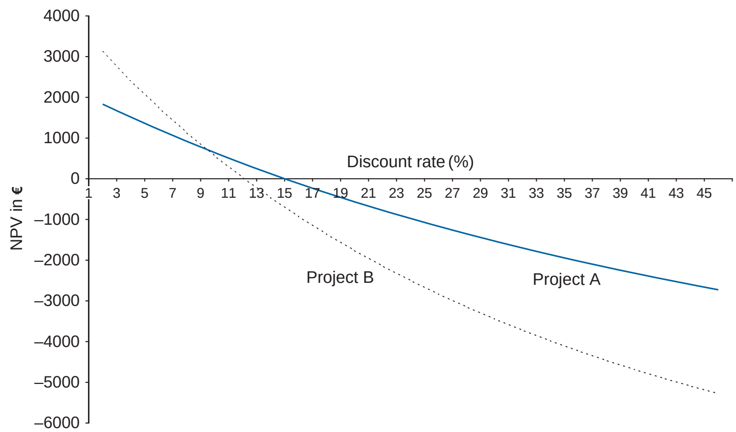
The same decision obtained by comparing the incremental IRR with the required rate of return could also be obtained by:

- simply comparing the NPV of the two projects. The large-scale project has a higher NPV and is the preferred project according to the NPV rule; and
- estimating the incremental NPV. If it is positive, then the large-scale project is preferable. Vice versa, the smaller project is more attractive if the incremental NPV is negative.

In order to understand the **timing problem**, consider two projects with the same initial amount (ergo, no problem of scale). Project *A* is a marketing campaign that could push the sales of existing products. The cash inflows are immediate but disappear progressively. This can be defined as the "short-sighted" project. Project *B* is a new product development with big positive cash inflows expected at the end of the development process. This will be defined as the "far-sighted" project:

| | 0 | 1 | 2 | 3 | IRR |
|---|---|---|---|---|---|
| Project *A* (Marketing campaign) | −10,000 | 8,000 | 3,000 | 1,000 | 14% |
| Project *B* (New product development) | −10,000 | 0 | 2,000 | 11,500 | 11% |

According to the IRR method, project *A* is more attractive because it has a higher IRR (14% vs. 11%). The NPV pro le of the two alternatives is:




In the graph above, it can be seen that the NPV of Project *B* is higher if the discount rate is low, say below 8%. When the discount rate is low, *B* has the higher NPV; when the discount rate is high, *A* has the higher NPV. If the discount rate is above 8% then the NPV of project *A* exceeds that of Project *B*. The NPV of project *B* declines more rapidly than the NPV of project *A*. This occurs because the cash ows of *B* occur later.

In order to determine which project is more attractive, a comparison should be made between the NPVs of the two projects. The decision will then be a function of the discount rate.[1]

A naïve reliance on the IRR method can lead to decisions that favour investments with short-term payoffs. Perhaps this is one of the reasons behind the frequent criticism regarding managers of public corporations and their supposed “short-termism”.

1 *Alternatively, we could subtract the cash flows of A from the cash flows of B, and then calculate the incremental IRR: if the discount rate is lower than the incremental IRR we should accept project B.*

## Section 17.4
## SOME MORE FINANCIAL MATHEMATICS: INTEREST RATE AND YIELD TO MATURITY

### 1/ NOMINAL RATE OF RETURN AND YIELD TO MATURITY

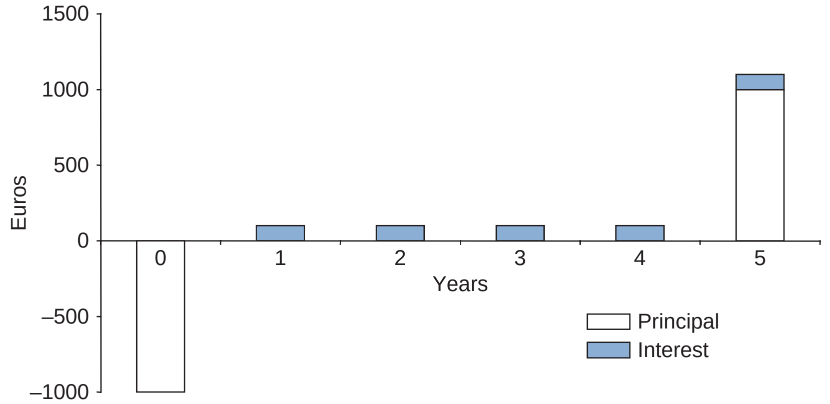
Having considered the yield to maturity, it is now important to examine interest rates; for example, on a loan that you wish to take out. Where does the interest rate t in this discussion?

Consider someone who wants to lend you €1000 today at 10% for ve years. 10% means 10 per cent **per year** and constitutes the **nominal rate of return** of your loan. This rate will be the basis for calculating interest, proportional to the time elapsed and the amount borrowed. Assume that you will pay interest annually.

The rst problem is how and when will you pay off the loan?

**Repayment terms constitute the method of amortisation of the loan.** Take the following example:

## (a) Bullet repayment



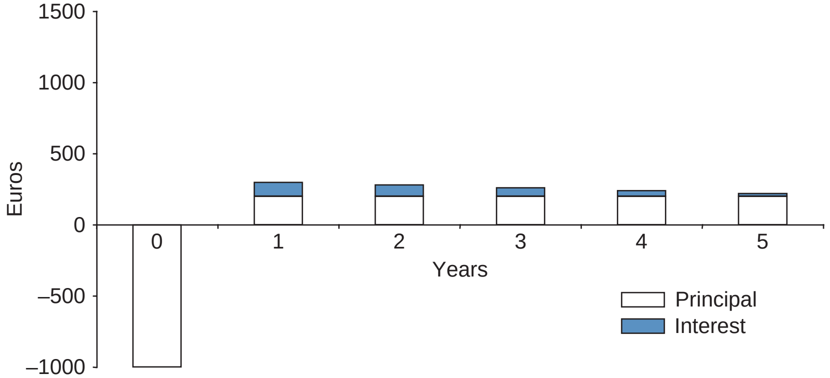

The entire loan is paid back at maturity.

The cash ow table would look like this:

| Period | Principal still due | Interest | Amortisation of principal | Annuity |
|---|---|---|---|---|
| 1 | 1000 | 100 | 0 | 100 |
| 2 | 1000 | 100 | 0 | 100 |
| 3 | 1000 | 100 | 0 | 100 |
| 4 | 1000 | 100 | 0 | 100 |
| 5 | 1000 | 100 | 1000 | 1100 |

**Total debt service** is the annual sum of interest and principal to be paid back. This is also called debt servicing at each due date.

## (b) Constant amortisation



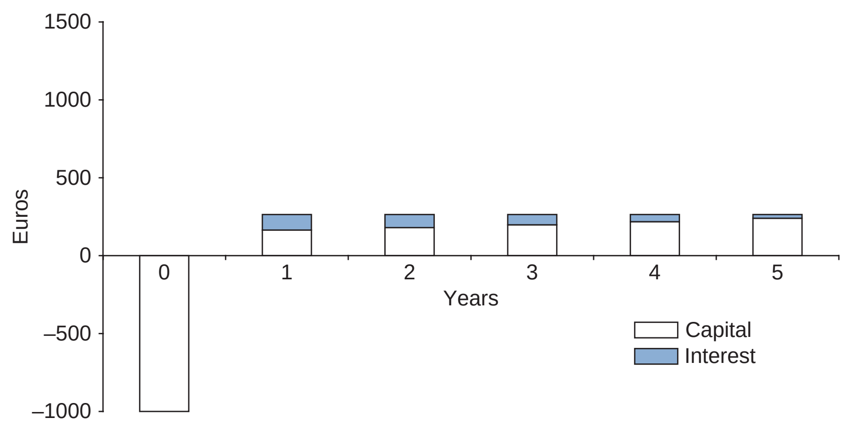

Each year, the borrower pays off a constant proportion of the principal, corresponding to $1/n$, where $n$ is the initial maturity of the loan.

The cash  ow table would look like this:

| Period | Principal still due | Interest | Amortisation of principal | Annuity |
|---|---|---|---|---|
| 1 | 1000 | 100 | 200 | 300 |
| 2 | 800 | 80 | 200 | 280 |
| 3 | 600 | 60 | 200 | 260 |
| 4 | 400 | 40 | 200 | 240 |
| 5 | 200 | 20 | 200 | 220 |

### (c) Equal instalments

The borrower may wish to pay off his loan by constant annuities, i.e. allocate a constant sum to interest and amortisation payments.




In the above cases, the borrower paid off either a constant sum in interest or a declining sum in interest. The principal was paid off in equal instalments.

Based on the discounting method described previously, consider a constant annuity *A*, such that the sum of the  ve discounted annuities is equal to the present value of the principal, or €1000:

$$1000 = \frac{A}{1.10} + \frac{A}{(1.10)^2} + \ldots + \frac{A}{(1.10)^5}$$

This means that **the NPV of the 10% loan is nil; in other words, the 10% nominal rate of interest is also the internal rate of return of the loan**.

Using the formula from Chapter 16, the previous formula can be expressed as follows:

$$1000 = \frac{A}{0.10} \times \left(1 - \frac{1}{(1.10)^5}\right)$$

$A = €263.80$. Hence, the following repayment schedule:

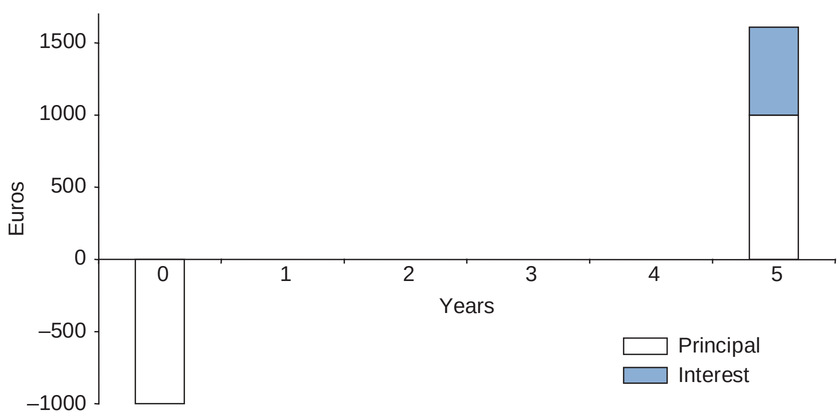
| Period | Principal still due | Interest | Amortisation of principal | Annuity |
|---|---|---|---|---|
| 1 | 1000 | 100 | 163.80 | 263.80 |
| 2 | 836.20 | 83.62 | 180.18 | 263.80 |
| 3 | 656.02 | 65.60 | 198.20 | 263.80 |
| 4 | 457.82 | 45.78 | 218.02 | 263.80 |
| 5 | 239.80 | 23.98 | 239.80 | 263.80 |

In this case, the interest for each period is indeed equivalent to 10% of the remaining principal (i.e., the nominal rate of return) and the loan is fully paid off in the  fth year. Internal rate of return and nominal rate of interest are identical, as calculation is on an annual basis and the repayment of principal coincides with the payment of interest.

Regardless of which side of the loan you are on, both work the same way. We start with invested (or borrowed) capital, which produces income (or incurs interest costs) at the end of each period. Eventually, the loan is then either paid back (leading to a decline in future revenues or in interest to be paid) or held on to, thus producing a constant  ow of income (or a constant cost of interest).

### (d) Interest and principal both paid when the loan matures

In this case, the borrower pays nothing until the loan matures.




The sum that the borrower will have to pay at maturity is none other than the future value of the sum borrowed, capitalised at the interest rate of the loan.

$$V = 1000 \times (1 + 10\%)^5 \text{ or } V = 1610.5$$

This is how the repayment schedule would look:

| Period | Principal and interest still due | Amortisation of principal | Interest payments | Annuity |
|---|---|---|---|---|
| 1 | 1100 | 0 | 0 | 0 |
| 2 | 1210 | 0 | 0 | 0 |
| 3 | 1331 | 0 | 0 | 0 |
| 4 | 1464.1 | 0 | 0 | 0 |
| 5 | 1610.51 | 1000 | 610.51 | 1610.51 |

**This is a zero-coupon loan.**

## 2/ EFFECTIVE ANNUAL RATE, NOMINAL RATES AND PROPORTIONAL RATES

This section will demonstrate that discounting has a much wider scope than might have appeared to be the case in the simple  nancial mathematics presented previously.

### (a) The concept of effective annual rate

What happens when interest is paid not once but several times per year?

Suppose that somebody lends you money at 10% but says (somewhere in the  ne print at the bottom of the page) that interest will have to be paid on a half-yearly basis. For example, suppose you borrowed €100 on 1 January and then had to pay €5 in interest on 1 July and €5 on 1 January of the following year, as well as the €100 in principal at the same date.

This is not the same as borrowing €100 and repaying €110 one year later. The nominal amount of interest may be the same ($5+5=10$), but the repayment schedule is not. In the  rst case, you will have to pay €5 on 1 July (just before leaving on summer holidays), which you could have kept until the following 1 January if using the second case. In the  rst case you pay €5, instead of investing it for six months as you could have done in the second.

As a result, the loan in the  rst case costs more than a loan at 10% with interest due annually. Its effective rate is not 10%, since interest is not being paid on the benchmark annual terms.

To avoid comparing apples and oranges, a  nancial of cer must take into account the effective date of disbursement. We know that one euro today is not the same as one euro tomorrow. Obviously, the  nancial of cer wants to postpone expenditure and accelerate receipts, thereby having the money work for him. So, naturally the repayment schedule matters when calculating the rate.

Which is the best approach to take? If the interest rate is 10%, with interest payable every six months, then the interest rate is 5% **for six months**. We then have to calculate **an effective annual rate** (and not for six months), which is our point of reference and our constant concern.

In our example, the lender receives €5 on 1 July which, compounded over six months, becomes $5 + (10\% \times 5)/2 = €5.25$ on the following 1 January, the date on which he receives the second €5 interest payment. So over one year, he will have received €10.25 in interest on a €100 investment.

**Therefore, the annual effective annual rate is 10.25%. This is the real cost of the loan, since the return for the lender is equal to the cost for the borrower.**

If the nominal rate ($r_a$) is to be paid $n$ times per year, then the effective annual rate ($t$) is obtained by compounding this nominal rate $n$ times:

$$(1+t) = (1 + r_a/n)^n$$

Formula for converting nominal rate into effective annual rate.

where $n$ is the number of interest payments in the year and $r_a/n$ the proportional rate during one period, or $t = (1 + r_a/n)^n - 1$.

In our example:

$$t = (1 + 10\%/2)^2 - 1 = 10.25\%.$$

The effective interest rate is thus 10.25%, while the nominal rate is 10%.

It should be common sense that an investment at 10% paying interest every six months produces a higher return at year-end than an investment paying interest annually. In the  rst case, interest is compounded after six months and thus produces interest on interest for the next six months. Obviously a loan on which interest is due every six months will cost more than one on which interest is charged annually.

It is essential to first calculate the effective annual rate before comparing investments (or loans) with different cash flow streams. The effective annual rate measures returns on the common basis of a year, thus making meaningful comparisons possible. This is not possible with nominal rates.

The table below gives the returns produced by an investment (a loan) at 10% at varying instalments:

| Interest compounding period | Initial sum | Sum after one year | Effective annual rate (%) |
|---|---|---|---|
| Annual | 100 | 110.000 | 10.000 |
| Half-year | 100 | 110.250 | 10.250 |
| Quarterly | 100 | 110.381 | 10.381 |
| Monthly | 100 | 110.471 | 10.471 |
| Bimonthly | 100 | 110.494 | 10.494 |
| Weekly | 100 | 110.506 | 10.506 |
| Daily | 100 | 110.516 | 10.516 |
| Continuous[2] | 100 | 110.517 | 10.517 |

Effective annual rate can be calculated on any time scale. For example, a  nancial of cer might wish to use continuous rates. This might mean, for example, a 10% rate producing €100, paid out evenly throughout the year on principal of €1000. The  nancial of cer will use the annual equivalent rate as his reference rate for this investment

2 *The formula for continuously compounded interest is:* $t = e^k - 1$

### (b) The concept of proportional rate

In our example of a loan at 10%, we would say that the 5% rate over six months is **proportional** to the 10% rate over one year. More generally, two rates are proportional if they are in the same proportion to each other as the periods to which they apply.

10% per year is proportional to 5% per half-year or 2.5% per quarter, but 5% half-yearly is not equivalent to 10% annually. **Effective annual rate and proportional rates are therefore two completely different concepts that should not be confused**.

**Proportional rates are of interest only when calculating the interest actually paid. In no way can they be evaluated with other proportional rates, as they are not comparable.**

Proportional rates serve only to simplify calculations, but they hide the true cost of a loan. Only the effective annual rate (10.25%/year) gives the true cost, unlike the proportional rate (10%/year).

When the time span between two interests payment dates is less than one year, the proportional rate is lower than the effective annual rate (10% is less than 10.25%).

To avoid error, use the effective annual rate.

As we will see, the bond markets can be misleading since they reason in terms of nominal rate of return: paper is sold above or below par value, the number of days used in calculating interest can vary, there could be original issue discounts, and so on. And, most importantly, on the secondary market, a bond's present value depends on fluctuations in market interest rates.

**In the rest of this book, unless otherwise specified, an interest rate or rate of return is assumed to be an effective annual rate.**

## SUMMARY

@ download

In this section we learned about the theoretical foundations of interest rates, which force financial managers to discount cash flows; i.e. to depreciate the flows in order to factor in the passage of time.

This led us to a definition of present value, the basic tool for valuing a financial investment, which must be compared to its market value. The difference between present value and the market value of an investment is net present value.

In a market in equilibrium the net present value of a financial investment is nil because it is equal to its present value.

As the value of an investment and the discount rate are fundamentally linked, we also looked at the concept of yield to maturity (which cancels out NPV). Making an investment is only worth it when the yield to maturity is equal to or greater than the investor's required return. At fair value, internal rate of return is identical to the required return rate. In other words, net present value is nil.

Internal rate of return should be handled with care, as it is based on the implicit assumption that cash flows will be reinvested at the same rate. It should only be relied on for an investment decision concerning a single asset and not for choosing from among several assets, whether they are financial (e.g. an investment) or industrial (e.g. a mine, a machine, etc.). NPV should be used for such decisions.

Finally, some financial mathematics helped us look at the link between the nominal interest rate and the yield to maturity of an operation. The nominal (annual) rate of a loan is the rate used to calculate interest in proportion to the period of the loan and the capital borrowed. However, one must use the yield to maturity, which may differ from the apparent nominal rate, when interest is not paid on an annual basis.

## QUESTIONS

@ quiz

1/ Why can't the internal rate of return be used for choosing between two investments?

2/ Does the interest rate depend on the terms of repayment of a loan or an investment?

3/ Does the interest rate depend on when cash flows occur ?

4/ What are proportional rates?

5/ What is internal rate of return?

6/ What are proportional rates used for? And internal rate of return?

7/ On the same loan, is the total amount of interest payable more if the loan is repaid in fixed annual instalments, by constant amortisation or on maturity?

8/ If you believe that interest rates are going to rise, would you be better off choosing loans that are repayable on maturity or in fixed annual instalments?

9/ If the purchase price of an investment is positive and all subsequent cash flows are positive, show how there can only be a single yield to maturity.

10/ Is it better to make a small percentage on a very large amount or a large percentage on a small amount? Does this bring to mind one of the rules explained in this chapter?

11/ A very high yield to maturity over a very short period is preferable to a yield to maturity that is 2% higher than the required rate of return over 10 years. True or false?

## EXERCISES

1/ What interest rate on an investment would turn 120 into 172.8 over two years? What is the yield to maturity? What is the proportional rate over three months?

2/ What is the terminal value on an initial investment of 100, if the investor is seeking a 14% yield to maturity after 7 years?

3/ For how many years will 100 have to be invested to get 174.9 and a yield to maturity of 15%?

4/ You invest €1000 today at 6% with interest paid on a half-yearly basis for 4 years. What is the yield to maturity of this investment? How much will you have at the end of the 4-year period?

5/ Investment A can be bought for 4 and will earn 1 per year over 6 years. What is the yield to maturity? Investment B costs 6 and earns 2 over 2 years, then 1.5 over 3 years.

What is the yield to maturity? Which investment would you rather have? Why? Do you need to know what the minimum rate of return is in order to make a decision?

6/ A company treasurer invests 100 for 18 months. The first bank he approaches offers to reinvest the funds at 0.8% per quarter, and the second bank, at 1.6% per half-year. Without actually doing the calculation, show how the first bank's offer would be the best option. What are the 2 yields to maturity?

7/ A company treasurer invests €10,000,000 on the monetary market for 24 days. He gets back €10,019,745. What is the rate of return over 24 days? What is the yield to maturity?

8/ Draw up a repayment schedule for a loan of 100, with a yield to maturity of 7% over 4 years, showing repayment in fixed annual instalments and constant amortisation.

9/ Draw up a repayment schedule for a loan of 400, with a yield to maturity of 6.5% over 7 years with repayment deferred for 2 years, showing repayment in fixed annual instalments and constant amortisation.

10/ A bond issued at 98% of the nominal value is repaid at maturity at 108% after 10 years. Annual interest paid to subscribers is 7% of the nominal value. What is the yield to maturity of this bond? And what if it had been issued at 101%? So what is the rule?

11/ What is the discounted cost for the issuer of the bond described in question 10 if we factor in a 0.35% placement commission, and annual management fee of 2.5% of the coupon, a closing fee of 0.6% of the amount paid, and an issue price of 98%.

12/ You sell your flat valued at €300,000 for a down payment of €100,000 and 20 monthly payments of €11,000. What is the monthly interest rate for this transaction? What is the yield to maturity?

13/ Calculate the yield to maturity of the following Investment, which can be purchased today for 1,000:

| Year | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Cash flow | 232 | 2088 | 232 | −232 | −927 |

## ANSWERS

*Questions*

1/ *Because it does not measure the value created.*
2/ *No.*
3/ *Yes.*
4/ *Rates that have a proportional relationship with the periods to which they relate.*
5/ *Rates that apply to different periods, but which transform the same sum in an identical manner over the same period.*
6/ *For calculating the interest that is paid out/earned. For calculating the yield to maturity.*
7/ *On maturity, because the principal is lent in full over the whole period.*
8/ *On maturity, so that you can take advantage for as long as possible of a low interest rate on the maximum amount of principal outstanding.*
9/ *At a discount rate equal to the yield to maturity, the present value of future cash flows is equal to the purchase price of the investment. If the discount rate increases, present*

*value will drop and will never again be equal to the market price of the investment. If the discount rate decreases, present value will rise and will never again be equal to the market price of the investment. Accordingly, there is only a single yield to maturity.*

10/ *A small percentage on a very large amount. NPV is preferable to yield to maturity.*

11/ *False, because an investment with an acceptable yield to maturity over a long period creates more value than an investment with a very high yield to maturity but which is of little significance given the short period of the Investment.*

*Exercises*

1/ *44% over 2 years. 20%. 5% over 3 months.*

2/ *250.*

3/ *4 years.*

4/ *6.09 %, €1266.8.*

5/ *13%, 13.8%, a choice between these two securities cannot be based on yield to maturity. Only NPV can be relied on. Yes, you have to know what the required rate of return is.*

6/ *As the rates are proportional (0.8% over 3 months and 1.6% over 6 months), the first offer is better, since interest is capitalised after 3 months and not 6. 3.24% and 3.23%*

7/ *0.1975% over 24 months, 3.05%.*

8/ *Fixed annual instalments of 29.53, constant amortisation of 25/year and interest of 7, 5.25, 5 and 1.75.*

9/ *Fixed annual instalments of 109.2, constant amortisation of 90.74/year and interest of 29.5, 23.6, 17.7, 11.8 and 5.9.*

10/ *7.85% (don't forget interest for year 10), 7.42%, value and rates vary in opposite directions.*

11/ *8.12%.*

12/ *0.925% 11.7%.*

13/ *There are 2: 15.1% and 48.3%*

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